On the threshold of the SIS epidemic model with random perturbation

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Introduction

Different models exist to study the progress of an epidemic in a population: SIR, SIRS, SIS etc.

- S $\rightarrow$ Susceptible
- I $\rightarrow$ Infected

Example : SIS models infections such us common flu

Deterministic SIS model

$$
\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt, \end{cases} \quad (1)
$$

where :

- $\mu \to$ birth rate (coinciding with death rate)
- $\beta \to$ infection coefficient
- $\gamma \to$ recovery rate

Constant population $\Rightarrow$ S + I = 1

Stochastic SIS model

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt, \end{cases}$$

$$\beta \ \rightsquigarrow \ \beta + \sigma dB$$

$$\begin{cases} dS = (\mu - \mu S - \beta SI + \gamma I)\, dt - \sigma SIdB, \\ dI = (-(\mu + \gamma)I + \beta SI)\, dt + \sigma SIdB, \end{cases} \tag{2}$$

Recently (2011), Gray et al. considered the precedent stochastic model (2)

With initial conditions (S_0, I_0) in the set $\Delta = \left\{x \in \mathbb{R}^2_+;\ x_1 + x_2 = 1\right\}$.
To solve the system 2 Gray et al. [2] proved that is sufficient to study the following EDS :

$$dI = \left(-(\mu+\gamma)I + \beta I(1-I)\right)dt + \sigma I(1-I)dB. \tag{3}$$

Definition

$\mathcal{R}_{\mathcal{S}}$ determines the dynamics of the SIS system.
It depends on the system parameters.
$\mathcal{R}_{\mathcal{S}} = f(\beta, \gamma, \mu, ...)$

1. persistence
2. exctinction

In deterministic case $\rightarrow \mathcal{R}_{\mathcal{S}} = \frac{\beta}{\mu+\gamma}$
In stochastic case $\rightarrow \mathcal{R}_{\mathcal{S}} = \frac{\beta}{\mu+\gamma+\frac{1}{2}\sigma^2}$

Gray et al. results on $\mathcal{R}_{\mathcal{S}}$ (case $\sigma^2 \leq \beta$)

- $\mathcal{R}_{\mathcal{S}} < 1 \longrightarrow$ extinction
- $\mathcal{R}_{\mathcal{S}} > 1 \longrightarrow$ persistence

$$\mathcal{R}_{\mathcal{S}} = 1 \longrightarrow \quad ?$$

Main results of this communication

Theorem

Let $(S_0, I_0) \in \Delta$*. If* $\mathcal{R}_{\mathcal{S}} < 1$ *then for all* κ *such that*

$$0 < \kappa < \frac{2\beta\left(1 - \mathcal{R}_{\mathcal{S}}\right)}{\sigma^2 \mathcal{R}_{\mathcal{S}}}, \tag{4}$$

the solution $I(t)$ *obeys*

$$\mathbb{E} I^{\kappa}(t) \leq I_0^{\kappa} e^{-\Lambda t}, \quad \text{where } \Lambda = -\kappa \left[\frac{\beta}{\mathcal{R}_{\mathcal{S}}} \left(\mathcal{R}_{\mathcal{S}} - 1\right) + \frac{\kappa}{2}\sigma^2 \right] > 0. \tag{5}$$

Then the disease-free equilibrium state E_0 *is* κ*-th moment exponentially stable.*

Theorem

Let $(S_0, I_0) \in \Delta$. *If* $\mathcal{R}_{\mathcal{S}} = 1$ *then for all* $\eta > 0$ *and* $\varepsilon > 0$, *we have*

$$\lim_{I_0 \to 0} \mathbb{P}\left(\sup_{0 \le t \le \eta} I(t) > \varepsilon \right) = 0, \tag{6}$$

that is, the disease-free equilibrium state E_0 *is stable in probability.*

Theorem

For any initial values $(S_0, I_0) \in \Delta$*, if* $\mathcal{R}_{\mathcal{S}} = 1$ *then the solution of SDE (3) obeys*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t I_s ds = 0$$

Simulation

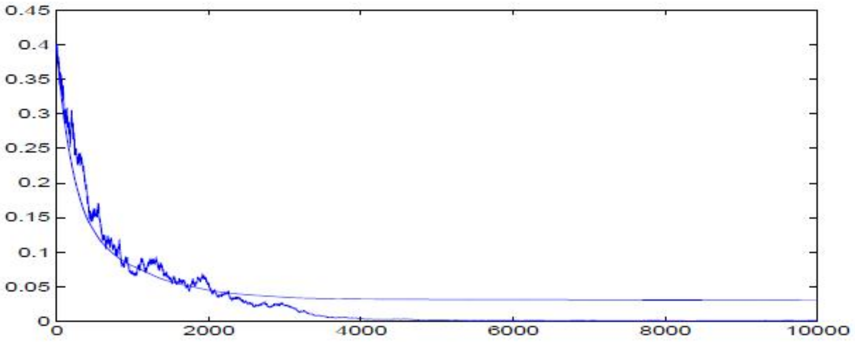

Figure 1 : Computer simulation of a single path of $I(t)$ for the SDE model (3) with initial condition $I_0 = 0.4$ and its corresponding deterministic model for the parameters: $\mu = 0.5$, $\beta = 0.902$, $\gamma = 0.4$, $\sigma = 0.2$, then $\mathcal{R}_0 > 1$ and $\mathcal{R}_{\mathcal{S}} = 1$.

The bibliography

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Thank you !